A Generalized Fuzzy Extension Principle and Its Application to Information Fusion

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Abstract—Zadeh’s extension principle (ZEP) is a fundamental concept in fuzzy set (FS) theory that enables crisp mathematical operation on FSs. A well-known shortcoming of ZEP is that the height of the output FS is determined by the lowest height of the input FSs. In this article, we introduce a generalized extension principle (GEP) that eliminates this weakness and provides flexibility and control over how membership values are mapped from input to output. Furthermore, we provide a computationally efficient point-based FS representation. In light of our new definition, we discuss two approaches to perform aggregation of FSs using the Choquet integral. The resultant integrals generalize prior work and lay a foundation for future extensions. Last, we demonstrate the extended integrals via a combination of synthetic and real-world examples.

Index Terms—extension principle, fuzzy set, fuzzy arithmetic, fuzzy integral, Choquet integral

I. INTRODUCTION

A fundamental concept in fuzzy set (FS) theory is Zadeh’s extension principle (ZEP) [1] that lays out an elegant framework to extend mathematical operations from classical crisp sets to FSs. ZEP has been used to tackle a wealth of challenges like computation on linguistic variables, calculus on linguistic probabilities [1], [2], arithmetic on fuzzy numbers or FSs [1], [3]–[13], in areas like computing with words [1], [14]–[16], information fusion [17], multi-criteria decision making [18], classification [19] and regression [20], to name a few.

At a high-level, ZEP extends the idea of element-to-element mapping in classical sets to FSs. However, ZEP achieves this by forcing the height of the output FSs to equal the lowest input set height. The conundrum is, this is not always semantically pleasing. For example, consider the case of three sets. Let the first set have a maximum membership (height) of 0.001 and let the other two sets be normal (height of 1). Whatever function we compute on these inputs will be restricted to a maximum membership degree of 0.001. However, many functions, e.g., the fuzzy integral, allow us to specify parameters that can express ideas like “input worth”. In such a case, if input one has little-to-no importance, why should it drastically impact the outcome? This example is expanded on later in this article, synthetically and with respect to real-world data. The point is, ZEP is a wonderful concept that pioneered mathematical operations for FSs [1]–[14], but it is currently too restrictive for many applications.

Some modifications to ZEP have been suggested. For example, Dubois proposed replacing the minimum operator in the mapping process with the product [21]. This change can yield an output whose height varies based on the input heights; however, the maximum height is limited to the lowest of the inputs heights still. Jain et al. proposed to use the probabilistic sum rather than the supremum [22], which may result in outputs that are probabilistic in nature rather than fuzzy and it can have an output with membership values higher than the maximum of the input membership values. Cooman et al. [13] proposed an extension that replaces the min intersection operator with the t-norm. Since no t-norm can attain greater value than min, their extension still has the same limitation as the ZEP for FSs with unequal heights.

Recently, other attempts have been made at the application level to mitigate the weaknesses of ZEP for specific algorithmic operations. One notable example is the fuzzy integral (FI), which is a family of nonlinear, monotonic aggregation operators, including the Choquet integral (ChI) and the Sugeno integral. In [23], Anderson et al. proposed a new definition for the FI of FSs, called non-direct fuzzy integral (NDFI) that, as the name implies, is not consistent with ZEP. The major difference between ZEP-based FI and NDFI is that the former applies aggregation on the elements (across the y domain) whereas the latter performs aggregation on the memberships (across the x domain), thus deviating from the definition of the FI. While this approach remedies the height challenge, it created another issue: convex sums of convex FSs result in non-convex FSs. In another work, Havens et al. introduced the shape preserving fuzzy integral (SPFI) [17] to preserve the shape of input FSs in the output. In SPFI, the FSs are first scaled to be normal (height of 1). Next, the FI is computed using ZEP and the result is re-scaled to make the final height equal to the FI of the heights of the input FSs. Such scaling is neither part of nor defined by the extension principle.

In this paper, we propose a new definition of the extension principle, called the generalized extension principle (GEP). GEP replaces the minimum operator in ZEP with a general function—called a membership mapping function—and specifies its properties. The choice of this function impacts the shape as well as the height—which can range from the minimum to the maximum of the heights of input FSs of the output. Thus, GEP provides users freedom to pick application-
specific appropriate function to characterize how the membership degrees be mapped from input to output. We also provide an efficient GEP method, by showing that it is sufficient to represent a FS with those points only on the membership space \( \mathcal{X} \). We also provide an efficient GEP representation, (iii) we propose two light of GEP and we propose different ways to accomplish it. In summary, our contributions are as follows: (i) we propose introducing notations as well as to understand the theory and provide an efficient GEP representation, (iii) we propose different ways to accomplish it.

### Definition 1. (Fuzzy set)

Let \( \mathcal{X} \) be a universe of discourse. An FS \( A \) on \( \mathcal{X} \) is a set of ordered pairs, \( A = \{(x, m_A(x)) | x \in \mathcal{X}\} \), where \( m_A : \mathcal{X} \rightarrow [0,1] \) is the membership function that maps \( \mathcal{X} \) to the membership space \( M \).

We denote \( \mathcal{F}(X) \) as the space of all fuzzy sets in the universe \( X \).

#### Definition 2. (Support)

The support of an FS \( A \), \( S(A) \), is all elements in \( X \) for which the membership is non-zero, \( S(A) = \{x \in \mathcal{X} | m_A(x) > 0\} \).

#### Definition 3. (\( \alpha \)-level set)

The crisp set of elements which belong to FS \( A \) at least to the degree \( \alpha \) is \( \alpha A = \{x \in \mathcal{X} \mid m_A(x) \geq \alpha\} \).

#### Definition 4. (Convex set)

An FS \( A \) is convex if \( m_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(m_A(x_1), m_A(x_2)) \), \( x_1, x_2 \in \mathcal{X}, \lambda \in [0,1] \). Alternatively, an FS is convex if all \( \alpha \)-level sets are convex.

#### Definition 5. (Fuzzy Number)

A fuzzy number normal FS defined on the real line \( \mathbb{R} \) such that \( m_A(x) \) is piecewise continuous.

### A. Fuzzy Measure and Fuzzy Integral

The FI is a class of integrals that have been primarily used to combine information from multiple sources. The major difference between the classical integral and the FI is that the latter can incorporate the interaction among sources. The FI aggregates evidence from sources with respect to the utility of subsets of sources resulted from the interaction of sources. The relative worth of all subsets of sources are encoded in a capacity or the fuzzy measure (FM).

#### Definition 6. (Fuzzy measure)

Let \( P \) be the set of inputs, \( P = \{p_i; i = 1, 2, \ldots, N\} \). The FM, \( g : 2^P \rightarrow \mathbb{R}^+ \), is a function with the following properties; (i) (boundary condition) \( g(\emptyset) = 0 \), and (ii) (monotonicity) if \( E, F \subseteq P \), and \( E \subseteq F \), then \( g(E) \leq g(F) \).

#### Definition 7. (Sugeno \( \lambda \)-measure)

The Sugeno \( \lambda \)-measure, \( g_{\lambda} \), is a special case of the FM with the following additional property. If \( E, F \subseteq P \) and \( E \cap F = \emptyset \), then \( g_{\lambda}(E \cup F) = g_{\lambda}(E) + g_{\lambda}(F) + \lambda g_{\lambda}(E)g_{\lambda}(F) \). Sugeno showed that there exists one real-valued \( \lambda > 1 \) that satisfies \( \lambda + 1 = \Pi_{k=1}^{N} g_{\lambda}([p_k]) \).

In this article, we focus on the ChI, a type of FI.

#### Definition 8. (Choquet integral)

The Choquet integral of integrand \( h : P \rightarrow \mathbb{R} \) with respect to \( g \) on \( X \) is

\[
\int h \circ g = C_g(h) = \sum_{j=1}^{N} h_{\pi(j)}(g(E_{\pi(j)}) - g(E_{\pi(j-1)})),
\]

for \( E_{\pi(j)} = \{p_{\pi(1)}, \ldots, p_{\pi(j)}\} \), \( g(E_{\pi(0)}) = 0 \), and permutation \( \pi \) such that \( h_{\pi(1)} \geq h_{\pi(2)} \geq \ldots \geq h_{\pi(N)} \).

### III. ZADEH’S EXTENSION PRINCIPLE (ZEP)

While ZEP was introduced in 1965 by Zadeh, several extensions have been proposed to date [1], [24], [22]. The definition of ZEP is as follows.

#### Definition 9. (ZEP)

Let \( f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_r \rightarrow Z \) be a mapping function such that \( z = f(x_1, \ldots, x_r) \). Let \( A_i \in \mathcal{F}(X_i), i = 1, 2, \ldots, r \). According to ZEP, the function \( f \) induces another function \( \hat{f} : \bigcup_{i=1}^{r} \mathcal{F}(X_i) \rightarrow \mathcal{F}(Z) \) defined for \( r \) FSSs, \( A_1, \ldots, A_r \), that map to an FS \( B \in \mathcal{F}(Z) \) as

\[
\hat{f}(A_1, A_2, \ldots, A_r) = B = \{(z, m_B(z)) \mid z \in Z\},
\]

where

\[
z = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X_1 \times \cdots \times X_r,
\]

1Sometimes a normality condition is imposed such that \( g(P) = 1 \).

2Shorthand notation \( h_i = h(p_i) \) is used.
and $f^{-1}$ is the inverse of $f$.

As seen at (2) and (3), ZEP for any function $f$ consists of two parts: (i) computing the image $z$ from $x_i$s using the function $f$ in a manner similar to crisp sets, and (ii) calculating the membership value at each $y$, which takes the supremum of the minimum of all membership elements of the inverse image of $y$. Notice that while finding a mapping of FSs under a function $f$, we use $f$ only for elements and a different function (supremum of minimum) for the membership degree.

The next theorem provides an efficient way to compute ZEP. It states that the $\alpha$-levels of the FS obtained by ZEP coincide with the images of the $\alpha$ by the crisp function [25], [26].

**Lemma 1.** Let $f : X_1 \times \cdots \times X_r \rightarrow Z$ be a continuous function and let $A_i$ be a fuzzy subset of $X_i$ for $i = 1, \ldots, r$. Then, for all $\alpha \in [0, 1]$, the following equality holds

$$\alpha f(A_1, \ldots, A_r) = f(\alpha A_1, \ldots, \alpha A_r).$$

Interested readers may refer to [25], [26] for mathematical proof of this lemma.

Several alternatives of ZEP have been developed. Jain et al. [22] proposed replacing the supremum with the probabilistic sum, $P(u, v) = u + v - uv$ (we refer to this as JEP). Dubois and Prade argued that this EP is more probabilistic than fuzzy. Another concern from a computational perspective arises from the fact that the probabilistic sum includes $(2^N - 1)$ terms for $N$ parameters. Even more serious concern is the output always degenerates to binary values, i.e., 0 or 1, for inputs with continuous FSs, resulting in a complete loss of fuzzy information. This is because the probabilistic sum of an infinite number of non-negative real-valued number equals to 1. Dubois and Prade [21] proposed an alternative EP that just replaces the min with product (we referred to this EP as DEP). Note that the $\alpha$-cut representation in Theorem 1 is only valid for ZEP and does not hold for other variants of EP, namely DEP and JEP. Of all these variants, the classical ZEP is the most common and widely used in the literature.

We compare these different variants of the EP for the arithmetic sum operation, $f : X_1 \times X_2 \rightarrow \mathbb{R}$, with $X_1 = X_2 = [0, 1]$ and $f(x_1, x_2) = x_1 + x_2$, using the following two examples.

**Example 1.** Figure 1(a) shows the input FSs, which consist of a subnormal triangular FS,

$$A_1(x_1) = \begin{cases} \frac{x_1}{0.1} & 0 < x_1 \leq 0.1 \\ 0.2 - x_1 & 0.1 < x_1 < 0.2 \\ 0 & \text{else} \end{cases}$$

with height 0.1 and a normal triangular FS,

$$A_2(x_2) = \begin{cases} \frac{x_1 - 0.5}{0.1} & 0.5 < x_1 \leq 0.6 \\ 0.7 - x_1 & 0.6 < x_1 < 0.7 \\ 0 & \text{else} \end{cases}$$

with height 1. The resultant FSs computed using ZEP, DEP, and JEP, shown in Fig. 1(b), have heights 0.1 (minimum of input FS heights), 0.1 (product of input FS heights), and 1, respectively.

**Example 2.** In this example, we consider two subnormal rectangular FS inputs as follows,

$$A_1(x_1) = \begin{cases} 0.2 & 0 \leq x_1 \leq 0.1 \\ 0 & \text{else} \end{cases}$$

and

$$A_2(x_2) = \begin{cases} 0.3 & 0.4 \leq x_2 \leq 0.5 \\ 0 & \text{else} \end{cases}$$

Figure 1(d) shows the outputs according to ZEP, DEP, and JEP that are also rectangular FSs with heights 0.2 (minimum of input FS heights), 0.06 (product of input FS heights), and 1.0, respectively.

It is obvious from these two examples that none of the existing EPs is suitable for all kinds of inputs, particularly those involving subnormal inputs. As seen in Fig. 1, the height of the output FS yielded by ZEP is driven solely by the input FS with lowest height. While DEP takes into account the heights of all input FSs to determine the height of the output FS, the output height can be lower than the minimum of input heights, as shown in Fig. 1(b); thus, this further worsens the height issue posed by ZEP. Furthermore, the JEP output is no longer fuzzy for continuous FSs and is not suitable for any of the cases presented here.

After examining the behavior of these EP variants, one question arises, is there a function that is suitable for all applications and inputs and yet it yields a desired behavior relative to that application? The answer is most likely not. As we will show in Section VI-A, different mapping functions serve different purposes, whichbacks the proposition that mapping function selection ideally rely on an application. Therefore, we introduce GEP to generalize the membership mapping function—liberating it from the min operation—which opens a set of possibilities for choosing a function appropriate for an application.

**IV. GENERALIZED EXTENSION PRINCIPLE (GEP)**

The proposed GEP consists of two functions: $f$ to map elements, and the supremum of $f_m$ to compute the membership degree. In this paper, we refer to $f_m$ as the membership mapping function. Unlike ZEP, $f_m$ is user defined and should satisfy properties that we describe next.

**Definition 10. (GEP)**

Let $f : X_1 \times \cdots \times X_r \rightarrow Z$ be a mapping function, i.e., $z = f(x_1, \ldots, x_r)$. Let $A_i \in \mathcal{F}(X_i), i = 1, 2, \ldots, r$. Furthermore, let $f_m : [0, 1] \rightarrow [0, 1]$ be a membership mapping function with the following properties: $f_m$ is non-decreasing, $f_m(0, \ldots, 0) = 0$, and $f_m(1, \ldots, 1) = 1$. We propose a fuzzy extension of $f$ such that the function $f$ induces another function $\hat{f} : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times \cdots \times \mathcal{F}(X_r) \rightarrow \mathcal{F}(Z)$
defined for \( r \) FSs, \( A_1, \ldots, A_r \), that map to an FS \( B \in \mathcal{F}(Z) \) as

\[
\hat{f}(A_1, \ldots, A_r) = B = \{(z, m_B(z)) | z \in Z\},
\]

where

\[
z = f(x_1, \ldots, x_r), \quad (x_1, \ldots, x_r) \in X_1 \times \cdots \times X_r,
\]

\[
m_B(z) = \begin{cases} \sup_{(x_1, \ldots, x_r) \in f^{-1}(z)} f_m(m_{A_1}(x_1), \ldots, m_{A_r}(x_r)) & \text{if } f^{-1}(z) \neq \emptyset \text{ and } x_i \in S(A_i), i = 1, 2, \ldots, r, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( f^{-1} \) is the inverse of \( f \).

The above definition ensures that the support of the output FS is obtained from the mapping of only those points that lie within the supports of the inputs. The non-decreasing properties of \( f_m \) guarantees that an increase in membership value does not decrease the output membership values. Furthermore, due to the boundary conditions, membership will always be non-negative and a crisp set will always be mapped to a crisp set regardless of the \( f_m \) used.

It is obvious that ZEP is just a special case of GEP when \( f_m \) is the minimum operation. Now, we illustrate GEP with two examples for different \( f_m \)s, including the minimum. The first example computes the square root for a single variable; the second example shows the arithmetic sum operation for two variables.

**Example 3.** Suppose a discrete FS on \( X = \mathbb{R} \) is

\[
A = \frac{0.1}{1} + \frac{0.5}{4} + \frac{1.0}{9}.
\]

We want to compute the square root, \( f(x) = \sqrt{x} \). Let \( \hat{f} \) be a GEP extension of \( f \) and \( B = \hat{f}(A) \). We chose the following two functions for \( f_m \), both of which are valid according to Def. 10: (i) \( f_m(m_A) = \min(m_A) \), where \( f_m \) is the same as in ZEP; and (ii) \( f_m(m_A) = \sqrt{m_A} \), where \( f_m \) is the same as \( f \). The output FSs based on these two \( f_m \)s are: (i) \( B = \frac{0.1}{1} + \frac{0.5}{4} + \frac{1.0}{9} \) and (ii) \( B = \frac{0.32}{1} + \frac{0.71}{4} + \frac{1.0}{9} \), respectively.

When \( f_m \) is the same as in ZEP, the degree of membership remains unchanged between the input FS and output FS for a univariate case due to \( f_m(m_A) = \min(m_A) = m_A \). However, this membership degree can be altered with the use of a different function, e.g., \( \sqrt{m_A} \), as shown in Fig. 2. This example also demonstrates the flexibility of GEP that permits the use of same function both for mapping of the elements and for computing of the membership degree—as long as it satisfies the properties outlined in Def. 10.

**Example 4.** Herein, we repeat Examples 1 and 2 in Section III to illustrate the arithmetic sum operation based on GEP for the following \( f_m \)'s

(i) \( \min, \quad f_m(m_{A_1}, m_{A_2}) = \min(m_{A_1}, m_{A_2}) \),

(ii) \( \text{mean, } \quad f_m(m_{A_1}, m_{A_2}) = \text{mean } (m_{A_1}, m_{A_2}) \),

(iii) \( \text{linear convex sum, } \quad f_m(m_{A_1}, m_{A_2}) = w_1 m_{A_1} + w_2 m_{A_2}, w_1 \geq 0, w_2 \geq 0, \text{ and } w_1 + w_2 = 1. \)

Figure 3 shows the resultant FSs. As we can see, \( f_m \) impacts the height and shape of the output FS and we can obtain an

\[3\text{We used } w_1 = 0.15 \text{ and } w_2 = 0.85 \text{ for these examples.}\]
output FS with height in the range between the minimum and the maximum of the input FS heights via a suitable selection of $f_m$.

GEP in Def. 10 provides the theory for arithmetic operation on FSs, but it does not lend itself to convenient calculation. While Theorem 1 provides a framework for efficient computation of ZEP through $\alpha$-cut. Theorem 1 is only valid when $f_m$ is the min. Next, we formulate an alternate representation of an FS that enables efficient computation of GEP.

A. Extension Principle Representation

Instead of using a support and membership function, a FS can be expressed as an infinite set of points, i.e., $A = \{(x, y) | x \in S(A) \text{ and } 0 \leq y \leq m_A(x)\}$, where $(x, y)$ is a point in the FS. The FS can be decomposed vertically. Let the vertical cut of a FS at $\beta_A \in S(A)$ be defined as $y$-coordinates of points on the vertical cut, i.e.,

$$\beta_A[A] = \{y | 0 \leq y \leq m_A(\beta_A)\} = [0, m_A(\beta_A)],$$

which is an interval between 0 and $m_A(\beta_A)$. Accordingly,

$$A = \bigcup_{\beta_A \in S(A)} \beta_A \times \beta_A[A],$$

where $\times$ denotes the Cartesian product. The degree of membership at $\beta_A$ can be recovered from the $\beta$-cut as

$$m_A(\beta_A) = \max \left(\beta_A[A]\right).$$

Based on this point-based representation, we show in the next theorem that the FS obtained by GEP can equivalently be obtained by mapping points within FSs.

Theorem 2. Let $A_i = \{(x_{A_i}, y_{A_i})\}; i = 1, \ldots, r$, be $r$ FSs represented as infinite sets of points, $f$ be an arithmetic operation or relation on $X_1 \times \cdots \times X_r$, $f_m$ be a membership mapping function, and $f$ be a fuzzy extension of $f$ according to GEP that maps FSs $A_i$s to an FS $B$, i.e., $B = f(A_1, \ldots, A_r)$. Then $B$ can also be represented as a set of points as $B = \{(x, y) | x = f(x_{A_1}, \ldots, x_{A_r}), y_B = f_m(y_{A_1}, \ldots, y_{A_r}), \forall (x_{A_i}, y_{A_i}) \in A_i, \text{ for } i = 1, \ldots, r\}$.

Proof. Without loss of generality, we prove this theorem for $r = 2$. The $\beta$-cut of $A_1$ at $\beta_{A_1} \in S(A_1)$ is

$$\beta_{A_1}[A_1] = [0, m_{A_1}(\beta_{A_1})],$$

and the $\beta$-cut of $A_2$ at $\beta_{A_2} \in S(A_2)$ is

$$\beta_{A_2}[A_2] = [0, m_{A_2}(\beta_{A_2})],$$

where $m_{A_1}(\beta_{A_1})$ is the upper bound of the $\beta$-cut $\beta_{A_1}[A_1]$ as well as the membership degree of $\beta_{A_1}$ at $x_i = \beta_{A_1}$ in the FS $A_i$. Let $\beta_B = f(\beta_{A_1}, \beta_{A_2})$. According to Def. 10, $\beta_B \in S(B)$ and is a constant. Then the $\beta$-cut of $B$ at $\beta_B$ is $\beta_B[B] = f_m(\beta_{A_1}[A_1], \beta_{A_2}[A_2]).$ Since $f_m$ is non-decreasing, and $f_m(0, 0) = 0$, $\beta_B[B]$ is an interval

$$\beta_B[B] = [0, f_m(m_{A_1}(\beta_{A_1}), m_{A_2}(\beta_{A_2}))].$$

Suppose there are $N - 1$ other combinations of $\beta_{A_{ij}} \in S(A_1)$ and $\beta_{A_{j2}} \in S(A_2)$ that map to the same $\beta_B = f(\beta_{A_{i1}}, \beta_{A_{j2}})$. Then using the similar analysis of $\beta_B[B]$, we get

$$\beta_B[j][B] = [0, f_m(m_{A_1}(\beta_{A_{i1}}), m_{A_2}(\beta_{A_{j2}}))], j = 2, 3, \ldots, N.$$  

The vertical cut at $\beta_B$ of the FS $B$ is the union of all resultant vertical lines at $\beta_B$, i.e., $\beta_B[B] = \bigcup_{\beta_B[j][B]} [0, \sup f_m(m_{A_1}(\beta_{A_{i1}}), m_{A_2}(\beta_{A_{j2}}))].$ As a result, the membership value of the output FS $B$, at $\beta_B$ is,

$$m_B(\beta_B) = \sup f_m(m_{A_1}(\beta_{A_{i1}}), m_{A_2}(\beta_{A_{j2}})),$$

where $\beta_{A_{i1}} \in S(A_1), \beta_{A_{j2}} \in S(A_2)$, and $f(\beta_{A_{i1}}, \beta_{A_{j2}}) = \beta_B, j = 1, \ldots, N$, which is identical to the expression in the upper branch of Eq. (5). Since this representation of an FS considers only those points in the support of FSs with membership value greater than zero, therefore $m_B(\beta_B) = 0$ when $\beta_B = f(\beta_{A_{i1}}, \beta_{A_{j2}}), \beta_{A_{ik}} \notin S(A_1) \text{ or } \beta_{A_{jk}} \notin S(B);$ which is equivalent to the second branch of Eq. (5). Thus, concluding our proof.

While this theorem shows that the FS $B$ can be obtained through pointwise computation of GEP, it does not limit the computation just only to points. For computational convenience, one can decompose a FS in different ways, e.g., vertical or horizontal lines, and then adopt appropriate method for the computation (e.g., linewise for vertical decomposition). Next, we provide a theorem that uses vertical decomposition to prove that a FS can be represented with just points only on
the membership function, thus providing the framework for efficient computation of the GEP.

**Theorem 3.** Let $A_i = (X_i, m_{A_i})$, $i = 1, \ldots, r$, be $r$ FSs, $f$ be an arithmetic operation or relation on $X_1 \times \cdots \times X_r$, $f_m$ be a membership mapping function, and $\bar{f}$ be a fuzzy extension of $f$ according to GEP that maps FSs $A_i$s to an FS $B$, i.e., $B = \bar{f}(A_1, \ldots, A_r)$. Let the points on the membership functions of $A_i$s be mapped via $\bar{f}$ to a set of points $C$, i.e., $C = \{ (x_C, y_C) \} = \{ (f(x_1, x_2, \ldots, x_N), f_m(m_{A_1}, m_{A_2}, \ldots, m_{A_N})) \}, \forall (x_1, m_{A_1}) \in A_i$, where $m_{A_i}(x_i)$ is the membership value of $A_i$ at $x_i$. Then, the degree of membership of $A$ at $\beta_B$ is $m_B(\beta_B) = \arg \sup_{y_C}(x_C, y_C), \forall (x_C, y_C) \in C$ and $x_C = \beta_B$.

**Proof.** Without loss of generality, we prove this theorem for two input FSs. Let $A_1$ and $A_2$ be two input FSs. It follows from Theorem 2 that the membership degree at any vertical cut of the output FS $B$, $\beta_B$, can be computed using only the membership values of the input FSs $A_i$s,

$$m_B(\beta_B) = \sup f_m(m_{A_1}(\beta_{A_1}), m_{A_2}(\beta_{A_2})), $$

where $\beta_{A_1} \in S(A_1), \beta_{A_2} \in S(A_2)$ and $f(\beta_{A_2}, \beta_{A_2}) = \beta_B, j = 1, \ldots, N$. Therefore, if $C$ is the image of all points on the membership functions of $A_i$s under the mapping of $f$, then

$$m_B(\beta_B) = \arg \sup_{y_C}(x_C, y_C), \forall (x_C, y_C) \in C \text{ and } x_C = \beta_B,$$

which concludes the proof.

**Remark 1.** The above method provides a general framework for GEP computation via $\beta$-cuts in support of a FS, which works irrespective of the shape of the inputs and can be applied to any type of FS, e.g., convex, non-convex, subnormal, or normal FSs. However, depending on the application and the membership function used, there can be another form of decomposition that is more efficient. A noteworthy example is the $\alpha$-cut based computation of FNs with the minimum membership mapping function.

The membership function of a continuous FS is comprised of an infinite number of points; hence, the computation of GEP based on Theorem 3 is still intractable. However, an efficient approximate computation (or exact, depending on the shape, function, and type of inputs) can be carried out by representing the FSs with segments via discretization of the input space, $x_1$. We propose one such method in Algorithm 1. By changing the horizontal resolution, one can trade-off computational complexity and precision.

**Algorithm 1: GEP**

1. Input: FSs $A_i$, $i = 1, 2, \ldots, r$; arithmetic operation $f$; membership operation $f_m$; and horizontal resolution $s$
2. Let the points on membership function of $A_i$ at $\beta$-cuts at horizontal resolution of $r$ be $p_i = \{ (x_i, m_{A_i}(x_i)) \}$
3. Initialize $P = 0$
4. for $(x_1, m_{A_1}(x_1)) \in A_1$ do
5. for $(x_2, m_{A_2}(x_2)) \in A_2$ do
6. calculate $p = (x_C, y_C) = (f(x_1, 2, x_r), f_m(y_1, y_2))$
7. Quantize $x_C$ with a resolution of $s$.
8. Add $p$ to $P$, $P = \{ P, p \}$
9. Find unique $x_C$ in $P$. Let the set of those $x_C$s be $Q$
10. for $x_B \in Q$ do
11. Calculate the membership value at $x_B$, $m_B(x_B) = \max(y_C), \forall (x_C, y_C) \in P$ and $x_C = x_B$
12. Draw lines connecting consecutive points in $\{ (x_B, m_B(x_B)) \}$ yielding the output FS $B$.

While GEP affects all functional and relational mappings related to FSs, we illustrate GEP with respect to the ChI in the context of data/information fusion.

**V. THE Choquet INTEGRAL OF FSs VIA GEP**

We begin this section with a review of existing approaches for the ChI FS: (i) the ChI using ZEP, (ii) NDFI, and (iii) SPFI. Then we illustrate how the ChI can be computed using GEP, discussing two different ways.

Let $P$ be the set of inputs, $P = \{ p_i \}, i = 1, 2, \ldots, r$, and $H : P \rightarrow FS(\mathbb{R})$ be an FS-valued integrand defined on a real-valued line. The FS-valued evidence from source $p_i$ is...
denoted as \( H_i = \{(h_i, m_{H_i}(h_i))\} \), where \( h_i \in \mathbb{R} \). Let the ChI of FS-valued inputs w.r.t. an FM \( g \) be an FS \( B \),
\[
\int H \circ g = C_g(H) = B = \{(z, m_B(z))\}.
\]

A. ChI via the ZEP

According to ZEP, we can write \( z \) as
\[
z = C_g(h),
\]
where \( C_g(h) \) is computed using Eq. (1). The degree of membership of \( B \) at \( z \) is
\[
m_B(z) = \begin{cases} 
\sup_{(h_1, \ldots, h_r) \in C_g^{-1}(z)} \min[m_{H_1}(h_1), \ldots, m_{H_r}(h_r)] & \text{if } C_g^{-1}(z) \neq \emptyset \\
0 & \text{otherwise},
\end{cases}
\]
where \( C_g^{-1} \) is the inverse of \( C_g \). Naturally, this extension of the ChI possesses all of the weaknesses of ZEP, which has been discussed extensively in our previous works [17], [23].

B. ChI via NDFI

To alleviate the height issue with ZEP, Anderson et al. [23] proposed the NDFI. This FI applies the aggregation operation on the degree of membership rather than on the elements \( h_i \), therefore it is not a valid FI extension according to ZEP. The degree of membership of FS \( B \) at \( z \in \mathbb{R} \) is
\[
m_B(z) = C_g(m(z)),
\]
where \( m(z) = [m_{H_1}(z), \ldots, m_{H_r}(z)] \) is a vector of membership degrees of \( H_i \)s at \( z \).

C. ChI via SPFI

Havens et al. [17] introduced SPFI that preserves the shape of the input FSs. Let \( \text{Height}(H_i) \) be the height of FS \( H_i \). In SPFI, \( z \) is calculated in the same manner as in Eq. (1), and membership degree at \( z \) is defined as
\[
m_B(z) = b \left\{ \begin{array}{ll}
\sup_{(h_1, \ldots, h_r) \in C_g^{-1}(z)} \min[\text{Norm}(m_{H_1}(h_1)), \ldots, \text{Norm}(m_{H_r}(h_r))] & \text{if } C_g^{-1}(z) \neq \emptyset \\
0 & \text{otherwise},
\end{array} \right.
\]
where \( \text{Norm}(m_{H_i}) \) is the normalized membership degree so that the height of \( \text{Norm}(H_i) \) becomes 1, i.e.,
\[
\text{Norm}(m_{H_i}) = m_{H_i}/\text{Height}(H_i).
\]
The scaling factor \( b \) is calculated as the ChI of the heights of the input FSs w.r.t. the same FM \( g \) and the heights are sorted according to a user defined criterion, such as the center of mass. Let the ChI of \( H \) be denoted as \( C_{g_H}(H) \), then
\[
b = C_{g_H}(H) = \sum_{i=1}^{r} \text{Height}(H_{\pi_m}(i)) \left( g(A_{\pi_m(j)}) - g(A_{\pi_m(j-1)}) \right),
\]
where \( \pi_m \) is a permutation of \( P \). If the inputs are sorted according to the center of mass, then \( c_m(p_{\pi_m(1)}) \geq \cdots \geq c_m(p_{\pi_m(r)}) \) and \( A_{\pi_m}(j) = \{p_{\pi_m(1)}, \ldots, p_{\pi_m(r)}\} \), where \( c_m \) stands for the center of mass.

D. ChI via the GEP

Recall, GEP has two functions: the support function \( f \) and the membership mapping function \( f_m \). While there is a wide range functions that can be used as \( f_m \), herein we use the FI. Our rationales are as follows. The minimum operator used by Zadeh in ZEP is one of the FI operators. Furthermore, the FI provides a wealth of other possibilities ranging from minimum to maximum.

Let \( C_g \) and \( C_{g_m} \) be two ChIs corresponding to \( f \) and \( f_m \). Then, \( z \) is
\[
z = C_g(h),
\]
which is identical to ZEP and can be calculated using Eq. (1). Note that \( h_i \)s are sorted based on their values as in the discrete ChI for scalar-valued inputs.

The computation of the membership degree at \( z \), \( m_B(z) \), follows Eq. (5), where \( C_{g_m} \) is used as the membership mapping function. The ChI, \( C_{g_m} \), is defined as
\[
C_{g_m}(m) = \sum_{j=1}^{N} m_{H_{\pi_m(i)}}(g(E_{\pi_m(j)}) - g_m(E_{\pi_m(j-1)})),
\]
for \( E_{\pi_m(j)} = \{p_{\pi_m(1)}, \ldots, p_{\pi_m(j)}\} \), \( g(E_{\pi(0)}) = 0 \), and permutation \( \pi_m \) of \( P \) that determines how the inputs, \( p_i \)s, are sorted. Unlike \( C_g \) that has only one sorting order for given \( h_i \)s, \( C_{g_m} \) offers flexibility with regards to how \( \pi_m \) is selected to sort inputs provided that \( C_{g_m} \) satisfies the properties of the membership mapping function. Herein, we present two approaches to pick a valid \( \pi_m \). We remark that possible approaches may not be limited to only these two.

a) Sort by the degree of membership: In this case, \( \pi_m \) is selected such that it orders \( p_i \)s based on their membership values. Mathematically, \( m_{H_{\pi_m(1)}} \geq m_{H_{\pi_m(2)}} \geq \cdots \geq m_{H_{\pi_m(r)}} \).

We refer to this method hereafter as GEP-MFI.

b) Sort by FS properties: The inputs are sorted based on a criterion other than membership degree. This criteria can be set based on FS properties such as center of mass, height, and area of FSs or it can be an arbitrary sequence. For example, when center of mass is used as a criteria, the inputs will be sorted such that \( c_m(p_{\pi_m(1)}) \geq \cdots \geq c_m(p_{\pi_m(r)}) \), where \( c_m(p_i) \) is the center of mass of FS-valued input \( p_i \). Using this approach, we can obtain a shape-preserving FI when (i) FM \( g \) in \( C_g \) is also used in \( C_{g_m} \), i.e., \( g_m = g \) and (ii) inputs in \( C_{g_m} \) are sorted based on \( h_i \), i.e., \( \pi_m = \pi \). We refer to this FI as GEP-SPFI. While we do not have a mathematical proof whether GEP-SPFI and SPFI are the same or not (which could be part of future research), the empirical analysis via examples in Section VI-A show that they yield equivalent results.

The scaling of height in SPFI using the ChI w.r.t. \( g \) is equivalent to mapping the membership degree in GEP-SPFI using the ChI w.r.t. \( g \), provided that inputs are sorted in the same manner in both methods. This explains the identical results by SPFI and GEP-SPFI.

It is simple to show that GEP can perform the NDFI. Let
\[
f^{-1}(x) = \begin{cases} 
(x, \ldots, x) & \text{if } x_i \in S(A_i) \text{ and } x_1 = \cdots = x_N = x \\
\emptyset & \text{else}
\end{cases}
\]
and \( f_m = C_g = \int m \circ g_m \). Then \( \tilde{f} \), according to Def. 10 of GEP constitutes the NDFI. The function \( f \) that performs elements-to-elements is non-monotonic; therefore, it is not an FI and the NDFI is not fully compliant with GEP definition of the ChI.

VI. EXPERIMENTS AND RESULTS

In this section, synthetic experiments are used to study the proposed ideas in a controlled fashion, and skeletal age-at-death estimation in forensic anthropology is used to demonstrate the real-world applicability of our methods.

A. Synthetic Examples of the ChI using GEP

Herein, we illustrate the two proposed approaches to compute the ChI on FSs: (i) GEP-MFI and (ii) GEP-SPFI. Specifically, examples are provided and we compare them to existing methods, namely ZEP-ChI, NDFI, and SPFI. To this end, the following operators were selected: minimum, mean, and maximum. These operators are chosen in part because they are diverse (semantically), they conceptually represent the “middle” and “extreme ends” of the aggregation spectrum. As such, they help us gain insight into how the choice of operator impacts the outcome. Furthermore, we chose the same set of operators for \( g_m \) as \( C_g \), which is a user defined function and unique to GEP. We used center-of-mass for GEP-SPFI and SPFI, which has been shown to have good results in [17]. Next, we give examples based on convexity of the input FSs.

1) Convex FS inputs: Consider three triangular membership functions, denoted by vertices, \( A_1 = \{(0.0,0.0),(0.1,0.0),(0.3,0.0)\} \), \( A_2 = \{(0.4,0.0),(0.5,1.0),(0.8,0.0)\} \), and \( A_3 = \{(0.7,0.0),(0.9,0.7),(1.0,0.0)\} \) (see Fig. 4). Figures 5, 6, and 7 show the results for aggregation based on minimum, mean, and maximum operators, respectively. Since all the approaches, except NDFI, differ only in the type of membership mapping function used, the support of the output FSs are the same, as expected. However, the height and shape of the output FSs (in other words, membership values) varies depending on the membership function used.

In GEP-MFI, the height of the result is determined by the membership mapping function. This means that the height of the FS obtained by GEP will be the average of the heights of the integrand when \( f_m \) is the mean. Similarly, when \( f_m \) is the minimum operator, as in ZEP, the output height will also be the minimum of the input heights. However, this method does not preserve the shapes of the inputs in the output, e.g., the result can be a polygon for triangular FSs, as shown in Fig. 6(a). Nevertheless, through the use of a suitable aggregation operator for \( f_m \), GEP-MFI lets the user control the membership function of the result relative to the inputs.

Next, the support of NDFI differs from that of ZEP or GEP and it ranges from \( \emptyset \) (Fig. 5(b)) to the union of integrand supports (Fig. 7(b)). NDFI performs “in place” aggregation of the membership values. Therefore, when the ChI is the minimum, the membership degree of the output at a point \( z \) is the minimum of the membership degrees of the integrand FSs at that point, i.e., it will be zero if at least one of the input FSs is zero. As a result, NDFI yields an empty FS for minimum when the intersection of the integrand is an empty set. While this method deviates from the standard definition of the FI, it can find application where the main goal is to aggregate the levels of confidence of the inputs rather than values themselves.

As we see, both SPFI and GEP-SPFI produce identical results and the appearance of their output resembles that of the input. That is, if the inputs are triangular in shape, then the output will also be triangular, which is not necessarily the case for GEP-MFI. These methods can return one of the inputs in its exact form as the output via appropriate selection of the FM. For example, the maximum returns the right-most FS\((A_3)\) with the highest center-of-mass and the minimum operator returns the left-most FS\((A_1)\). These methods can be suitable for input selection or in applications where the membership degrees need to be weighted based on the relative position (or order) of the inputs defined by an FS sorting order.

Last, when the input FSs are convex, GEP-MFI, GEP-SPFI, and SPFI all produce convex outputs. On the other hand, NDFI usually produces non-convex output.

2) Non-convex FS inputs: Consider the following three FSs denoted by vertices, \( A_1 = \{(0.0,0.0),(0.1,1.0),(0.2,0.7),(0.25,1.0),(0.3,0.0)\} \), \( A_2 = \{(0.25,0.0),(0.35,1.0),(0.4,0.0)\} \), and \( A_3 = \{(0.6,0.0),(0.7,0.2),(0.8,0.1),(0.9,0.5),(1.0,0.0)\} \), shown in Fig. 8. Note, \( A_1 \) and \( A_3 \) are convex. We have almost the same observations for this case as we had for convex inputs. For example, the output for non-convex FS inputs for SPFI also coincides with GEP-SPFI. As shown in Figs. 10 and 11, the maximum membership for the minimum \( f_m \) is higher than the maximum \( f_m \). As expected, the output is non-convex for all ChI extensions.

Last, the ChI depends not only on the aggregation operator but also how the inputs are sorted. As such, GEP-MFI and GEP-SPFI exhibit completely distinctive behavior. It can be elicited from the observations made in the examples that none of the approaches is universally the best. The choice of a particular method as well as the membership mapping function depends on the problem in hand. For example, GEP-MFI is good for an application where the membership values are
Fig. 5: Illustration of the ChI of convex FSs with respect to the minimum aggregation function.

Fig. 6: Illustration of the ChI of convex FSs with respect to the mean aggregation function.

Fig. 7: Illustration of the ChI of convex FSs with respect to the maximum aggregation function.

Fig. 8: The non-convex FS inputs that are used in the examples in Sec. VI-A. The aggregation results based on GEP using the min, mean, and max FMs are shown in Figs. 9, 10, and 11 respectively.

B. Age-at-death estimation

We now provide a real-world example of estimating the age-at-death from human skeletal remains in forensic anthropology, which was first presented in [27]. Table II shows eight aging methods, each providing an interval estimation of age-at-death of skeletal remains. The bones used in these methods is assigned a quality between 0 and 1 by forensic anthropologists weighted based on their values whereas GEP-SPFI is good where membership values are aggregated not based on their values but on the ordering of the FSs as a whole. We remark that GEP does not limit the definition of the ChI to just the two techniques discussed and one could propose new ways appropriate for a given application.
based on the conditions of the samples. The reliability of each test, which is based on statistical tests in forensic anthropology, is shown in the last column. These reliability scores are used as the densities in the Sugeno $\lambda$-measure, $g_\lambda$.

Each aging method input is first transformed to a rectangular (or interval) FS with the interval being the estimated death-at-age and the height is the quality of the samples (see Figure 12(a)). Therefore, the width gives the uncertainty in the measurement and the height gives the membership of that measurement as given by the quality of the bone sample. The goal is to fuse these rectangular FSs to provide an aggregated FS with membership degree at each age-at-death.

Figure 12(b) shows the results of different FI based aggregation methods, namely GEP-MFI based on the minimum (GEP-MFI(min)), GEP-MFI based on the mean (GEP-MFI(mean)), and GEP-MFI based on the maximum (GEP-MFI(max)).

<table>
<thead>
<tr>
<th>Aging Method</th>
<th>Quality</th>
<th>Est. Age</th>
<th>Reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pubic Symphysis (PS)</td>
<td>0.6</td>
<td>35-39</td>
<td>0.57</td>
</tr>
<tr>
<td>Auricular Surface (AS)</td>
<td>0.8</td>
<td>35-39</td>
<td>0.72</td>
</tr>
<tr>
<td>Ectocranial Sutures-vault (ESv)</td>
<td>0.2</td>
<td>25-74</td>
<td>0.59</td>
</tr>
<tr>
<td>Ectocranial Sutures-lateral (ESl)</td>
<td>0.5</td>
<td>23-63</td>
<td>0.59</td>
</tr>
<tr>
<td>Sternal Rib Ends (SRE)</td>
<td>0.5</td>
<td>33-42</td>
<td>0.75</td>
</tr>
<tr>
<td>Endocranial Sutures (ES)</td>
<td>0.4</td>
<td>35-39</td>
<td>0.51</td>
</tr>
<tr>
<td>Proximal Humerus (PH)</td>
<td>0.3</td>
<td>37-86</td>
<td>0.44</td>
</tr>
<tr>
<td>Proximal Femur (PF)</td>
<td>0.7</td>
<td>25-76</td>
<td>0.56</td>
</tr>
</tbody>
</table>
MFI(min)), mean (GEP-MFI(mean)), and maximum (GEP-MFI(max)) membership mapping functions, GEP-SPFI and NDFI. All FI methods except NDFI produce rectangular FSs with an interval of 35 to 72 years with only difference being the height/membership degree. GEP-MFI(min) (which is also ZEP) has the lowest height, which is determined solely by the input FS with the lowest membership value (ESv). GEP-MFI(max) has the highest membership degree and GEP-SPFI resides somewhere between GEP-MFI(mean) and GEP-MFI(min). NDFI produces a unique output with the highest membership degree around 35 years where the methods agree the most and a lower confidence around that estimate.

The NDFI was designed specifically to solve this particular age-at-death estimation problem and hence is preferred to the standard FIs. In contrast to traditional FI methods that cause loss of age-specific information due to sorting, the NDFI performs age-at-death in-place aggregation, thus capturing agreement or disagreement among methods at each age. This NDFI also exposes the limitation of ZEP that restricts the membership mapping function to the minimum operator (or infimum for continuous FS), invalidating the use of the FI to aggregate membership value, which appears to be a better fit for this specific problem. The proposed extension replaces minimum with a more flexible monotonic function, broadening the set of operations that can be performed and better serve the purpose of an application while being in the realm of the extension principle.

VII. CONCLUSION AND FUTURE WORK

Herein, we proposed a new definition of the extension principle that eliminates the major weakness of ZEP for subnormal FS inputs. With the introduction of a membership mapping function, this theory provides a flexible and powerful means to perform algebraic and arithmetic operations with FSs. We introduced an alternate representation of a FS and showed that it is sufficient to consider only the points on the membership function surface of a FS. Based on this representation, we provided an algorithm that enables efficient computation through vertical decomposition, herein referred to as a $\beta$-cut. This method can be used for any types of FSs, whether convex or not. We illustrated GEP through examples of two arithmetic operations, square and sum.

The proposed theory impacts all areas of linguistic and mathematical functions and relations defined on FSs. Herein, we chose to explore the ChI for FSs, a mathematical function to aggregate information from multiple sources. Given a membership mapping function, we put forth two ways to calculate the membership degree for the ChI. We also showed that our proposed theory unifies different techniques for computing the FI, namely NDFI and SPFI, which was either fully inconsistent with ZEP or could not be obtained via direct application of ZEP. Furthermore, we compared and contrasted proposed approaches, as well as NDFI and SPFI, with examples.

We introduced herein an FS mapping method that in general applies to all types of inputs. However, further efficiency may be obtained for a particular type of FS, function, and/or membership mapping function. One such example is convex FS inputs with the minimum membership mapping function, for which the $\alpha$-cut method is computationally far more efficient. In the future, we will investigate and work on developing efficient algorithms for specific cases, such as non-decreasing functions and the mean and maximum mapping functions. We will also extend our previous work on real-valued inputs [28] to FSs for efficient computation and optimization of the ChI. Last, we will explore ways to learn the support and membership functions from data, when/where applicable, in our current era of machine learning.

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